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A MODIFIED DISCRETE ORDINATE APPROACH TO NONLINEAR CYLINDRICAL HEAT TRANSFER

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Abstract—Non-linear heat transfer between concentric cylinders is studied numerically based on the BGK-model equation. The numerical procedure adopted is a combination of the Willis integral iteration scheme and the discrete ordinate method. The results for constant collision frequency compare very well with other existing numerical results based on the integral equation approach. Cases with variable collision frequency are also considered and the results show that there are only slight changes with respect to the constant collision frequency cases. The method used in this paper can be applied to a wide class of problems.

NOMENCLATURE

b,	the ratio of the outer cylinder radius to the inner cylinder radius;
c,	the dimensionless molecular velocity vector;
c_z,	the z-component of c, see Fig. 1;
f,	the dimensionless molecular distribution function;
g, h,	modified distribution functions, see equation (7);
M,	the mass flux in the r-direction, see equation (12);
n,	the dimensionless number density;
n_2,	N_2/N_1;
N_1, N_2,	the "number densities" associated with the boundary conditions, see equation (5);
Q,	the heat flux in the r-direction, see equation (11);
Q_t,	the total heat flux, see equation (12);
r,	the radial distance, see Fig. 1;
s,	the polar component of c, see Fig. 1;
t,	the dimensionless temperature;
t_2,	T_2/T_1;
T_1, T_2,	the temperatures at the inner cylinder and the outer cylinder respectively;
x, y,	the transformed variables of r and ψ;
α,	the inverse Knudsen number, see equation (6);
ν,	the dimensionless collision frequency, see equation (6);
ψ,	the polar angle of c, see Fig. 1.

INTRODUCTION

THE NON-LINEAR kinetic equations have thus far not yielded to analytic attempts at their solution, at least not in any way that is widely applicable. However, non-linear problems can be solved by direct methods such as the Monte Carlo and discrete ordinate methods [1]. These methods have been successfully used not only in rarefied gas dynamics but also in related fields such as radiative transfer and neutron transport (e.g. see [2, 3].). The development of such techniques is thus of considerable utility.

In this paper we report the results of the application of a modification of the discrete ordinate method to the problem of heat transfer in a rarefied gas between concentric cylinders due to a large temperature difference. Although the problem itself is of no great interest per se it is useful for studying the effects of non-linearity and non-planar geometry, and also lends itself to experimental verification. To simplify the computational effort a model of the Boltzmann equation is used; however, there is no conceptual difficulty in extending the method to a full Boltzmann equation with cut-off intermolecular force potential. This problem has been previously treated by Anderson [16] who transformed the problem into non-linear integral equations and solved them using a sophisticated numerical method. Anderson's method, while a very useful numerical study, becomes increasingly complicated with the geometry and the governing kinetic equation. The method suggested here, on the other hand, is simple in concept and straightforward in application and can easily be extended to problems of complicated geometry and more exact model equations. The results for constant collision frequency achieved agree very well with Anderson's; we have, in addition, considered the variable collision frequency case.

DISCUSSION OF THE METHOD

For a monatomic neutral gas the molecular velocity distribution function in general depends on time, position and molecular velocity. In the discrete ordinate method, one tries to determine the distribution function over a set of suitably chosen sample points, instead of over all points, in velocity space. Moments of the distribution function are then obtained using numerical quadratures based on this set of points. The discretization process transforms the single transport equation with a varying parameter, i.e. the velocity c, into a hyperbolic system of coupled first order partial differential equations. If Cartesian co-ordinate systems are used, and if proper care is taken in regard to the collision integral, the set of discrete ordinates of the molecular velocity will appear only as constants. Usually one then solves the system of partial differential equations using iteration and a finite-difference approximation to handle the remaining variables (e.g. see [4, 5]). When curvilinear coordinate systems are used the transport equation in general will contain derivatives with respect to the velocity variables as well. In principle one may devise a finite difference scheme for such derivatives based on the set of discrete ordinates, although to our knowledge no calculations of this kind have been carried out so far.

For a hyperbolic system of first order partial differential equations there are usually two ways of carrying out the finite difference approximation; one is to use a set of rectangular nets and the other is to use the characteristic lines of the system [6]. Provided proper care is taken of the domain of dependence and the numerical stability consideration, both numerical procedures are convergent. All published results of rarefied gas dynamics problems treated by discrete ordinate methods have employed the first scheme. However, it seems that the second scheme will offer some computational advantage because we do not have to use a difference approximation for the derivatives along the characteristic lines as was done in [6]. We may first formally integrate this kind of transport equation in a special way, even before the application of discrete ordinate approximation, and then make suitable approximations.

Consider, for example, the problem of non-linear plane Couette flow with heat transfer as treated by Huang *et al.* [4], where the BGK-model equation was used. A typical discrete ordinate equation has the form of

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \alpha[G(u) - u] \tag{1}$$

where α is taken to be a constant. If we regard $G(u)$ as a known function we can integrate the above equation formally to obtain

$$u(x) = u(x_0)\exp\left[-\alpha(x - x_0)\right] + \alpha\int_{x_0}^{x}\mathrm{d}y\, G[u(y)]\exp\left[-\alpha(x - y)\right]. \tag{2}$$

If we write $\Delta x = x - x_0$, we may approximate the above equation by

$$u(x) = u(x_0)\exp\left[\alpha\,\Delta x\right] + \bar{G}(x)[1 - \exp(-\alpha\,\Delta x)] \tag{3}$$

where $\bar{G}(x)$ is a suitable "guess", usually some kind of mean value based on the previous iteration. Huang *et al.* approximated equation (1) by

$$\frac{u(x) - u(x_0)}{\Delta x} = \alpha\{\bar{G}(x) - \tfrac{1}{2}[u(x) + u(x_0)]\} \qquad (4)$$

i.e.

$$u(x) = \frac{(1 - \frac{1}{2}\alpha\Delta x)\, u(x_0) + \alpha\Delta x\bar{G}(x)}{1 + \frac{1}{2}\alpha\Delta x}.$$

Provided that $G(u) = \bar{G}(x) + O(\Delta x)$ in (x, x_0) it is easy to see that both equations (3) and (4) have a truncation error of $O(\Delta x)^2$ with respect to equation (2). Nevertheless, in practice equation (3) will be superior than equation (4) especially when $\alpha\Delta x$ is not sufficiently small. Bramlette [7] has carried out some calculations on the same problem as treated by Huang *et al.* [4]. His results agree very well with those of Huang with considerably less computational effort. For example, for Knudsen number equal to unity Bramlette used 64 points in the velocity space and 5 space steps across the physical space, while Huang used 144 points, and 50 steps respectively. There are about five iterations involved and the difference in successive iterates of the macroscopic variable is 10^{-4}. This is perhaps expected since equation (3) has a form similar to the Willis integral iteration scheme [8], while equation (4) is closer to the Knudsen iteration (e.g. see [8]). It is generally recognized that integral iteration is the better of the two.

We note that equation (1) is an ordinary differential equation. This is due to the simple geometry of the problem. However, similar conclusions can also be drawn for a general problem. In those cases, the common truncation errors of equations (3) and (4) will contain terms essentially due to the interpolation required to obtain the $u(x_0)$ term in equation (2) in addition to those introduced due to the approximation of $G(u)$.

To gain experience in the alternative approach we first applied it to the linearized cylindrical Couette flow problem [9]. The results checked well with other existing numerical solutions. In this paper we extend the treatment to the problem of nonlinear heat transfer between concentric cylinders, which was also treated by Anderson [10] using an integral equation method. As is the case for most discrete ordinate methods, extensions to nonlinear problems do not significantly complicate the problem. In fact we have found that in this case the nonlinear problem is actually more manageable than the linear problem.

FORMULATION OF THE PROBLEM

Consider the problem of steady heat transfer between two concentric cylinders. The inner cylinder with radius R_1 is maintained at a constant temperature T_1 while the outer cylinder with radius R_2 is maintained at T_2. For the boundary conditions at the walls we assume diffuse reflection with unity thermal accommodation coefficients, i.e. the distribution of particles emitted from the boundary point B is given by

$$F_B = N_B(2\pi RT_B)^{-\frac{3}{2}} \exp[-\xi^2/2RT_B] \qquad (5)$$

where R is the gas constant, ξ is the speed of the molecules, T_B is the wall temperature and N_B must be determined as part of the solution from the mass conservation law applied at the boundary. The choice of equation (5) is only for convenience and other boundary conditions may be used without introducing any complications. It should also be noted that our results can be applied directly to problems with diffuse boundary conditions but with incomplete accomodation inefficients.

Let us choose R_1, N_1, T_1, $(2RT_1)^{\frac{1}{2}}$ and $N_1(2RT_1)^{-\frac{3}{2}}$ as the scaling parameters for distance, number density, temperature, molecular velocity and the molecular distribution function, respectively. Although N_1 is an unknown quantity it can be related to the average number density, N_0, between the cylinders once the problem is solved. It is obvious that N_1 is linearly proportional to N_0, and while N_0 might appear to be a more logical choice the use of N_1 results in the simplest formulation. Denote the dimensionless variables of position, molecular velocity, number density, temperature and distribution function by $\boldsymbol{r} = (r, 0, z)$, $\boldsymbol{c} = (s, c_z)$, n, t and f, respectively (See Fig. 1 for the coordinate systems). The governing transport equation and boundary conditions can then be written as [17]

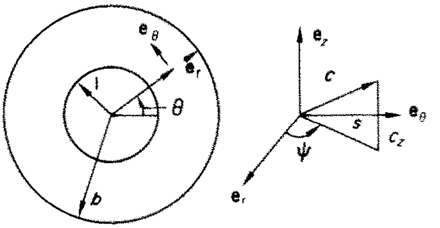

FIG. 1. Physical space and velocity space coordinates.

$$s\left[\cos\psi\frac{\partial f}{\partial r} - \frac{\sin\psi}{r}\frac{\partial f}{\partial\psi}\right] = \alpha\nu\,[F - f]$$
$$f(r = 1) = \pi^{-\frac{3}{2}}\exp[-c^2], \qquad |\psi| < \pi/2 \tag{6}$$
$$f(r = b) = n_2(\pi t_2)^{-\frac{3}{2}}\exp[-c^2/t_2], \qquad |\psi| > \pi/2$$

where $b = R_2/R_1$, $n_2 = N_2/N_1$, $t_2 = T_2/T_1$, $\alpha = \nu_0 R_1/\sqrt{(2RT_1)}$ and $F = n(\pi t)^{-\frac{3}{2}}\exp[-c^2/t]$. α may be regarded as the inverse of the Knudsen number. ν_0 is some reference collision frequency, and ν is the dimensionless collision frequency. In general we may take $\nu_0 = Pr\, p/\mu$ where p is the pressure, Pr the Prandtl number, and μ the viscosity coefficient [12]. If we regard Pr as constant and let μ vary as some power of temperature only, then $\nu = nT^{1-\sigma}$ where σ is the index of the power law of the viscosity. For future reference we note that, because of symmetry and the assumed boundary conditions, f depends only on r and $\boldsymbol{c}$ and is an even function of ψ. Furthermore, there is no net flow velocity.

To save computer storage space we introduce, following the procedure adopted by Huang and Hartley [4], the following two functions

$$g(r, s, \psi) = \pi e^{s^2}\int_{-\infty}^{\infty} \mathrm{d}c_z\, f$$
$$h(r, s, \psi) = 2\pi e^{s^2}\int_{-\infty}^{\infty} \mathrm{d}c_z c_z^2\, f \tag{7}$$

where the factor e^{s^2} is introduced to anticipate the later adoption of modified half-range Gauss–Hermite quadrature points [13] as the discrete ordinates in the s-coordinate. The governing equations for g and h are, respectively,

$$s\left[\cos\psi\frac{\partial g}{\partial r} - \frac{\sin\psi}{r}\frac{\partial g}{\partial\psi}\right] = \alpha\nu[G - g]$$
$$G = nt^{-1}\exp[-(t^{-1} - 1)s^2]$$
$$g(r = 1) = 1, \qquad |\psi| < \pi/2 \tag{8}$$
$$g(r = b) = n_2 t_2^{-1}\exp[-(t_2^{-1} - 1)s^2], \qquad |\psi| < \pi/2$$

and

$$s\left[\cos\psi\frac{\partial h}{\partial r} - \frac{\sin\psi}{r}\frac{\partial h}{\partial\psi}\right] = \alpha\nu[H - h]$$
$$H = n\exp[-(t^{-1} - 1)s^2] \tag{9}$$
$$h(r = 1) = 1, \qquad |\psi| > \pi/2$$
$$h(r = b) = n_2\exp[-(t_2^{-1} - 1)s^2], \qquad |\psi| > \pi/2.$$

In terms of g and h the density and temperature are defined by

$$\left.\begin{aligned} n &= \frac{2}{\pi}\int_0^{\pi} \mathrm{d}\psi\int_0^{\infty} \mathrm{d}s\, e^{-s^2} s g \\ nt &= \frac{4}{3\pi}\int_0^{\pi} \mathrm{d}\psi\int_0^{\infty} \mathrm{d}s\, e^{-s^2} s\left[\frac{h}{2} + s^2 g\right]. \end{aligned}\right\} \tag{10}$$

For this problem one would also be interested in the heat flux. If we scale the heat flux by $N_1 m(2RT_1)^{\frac{3}{2}}/2\pi$, where m is the mass of a gas molecule, the dimensionless heat flux Q is defined by

$$Q = 2\int_0^\pi d\psi \cos\psi \int_0^\infty ds\, e^{-s^2}\left[\frac{h}{2} + s^2 g\right] s^2. \quad (11)$$

We note that the total heat flux must be the same across any closed curve around the inner cylinder. This can be used as a check against numerical errors. We can also use the condition that there should be no net flow velocity as another check. These two conditions are expressed as

$$\left.\begin{aligned} Q_T &= rQ = \text{constant} \\ M &= \int_0^\pi d\psi \cos\psi \int_0^\infty ds\, e^{-s^2} s^2 g = 0 \end{aligned}\right. \quad (12)$$

where in the first equation we have used concentric circles as the closed curve.

For future reference we record the free-molecular solution here. It is easy to see that in this limit (i.e. $\alpha \to 0$) we have

$$\left.\begin{aligned} g &= 1, \quad |\psi| < \sin^{-1} 1/r \\ &= n_2 t_2^{-1} \exp\left[-(t_2^{-1} - 1)s^2\right], \\ &\qquad |\psi| > \sin^{-1} 1/r \\ h &= 1, \quad |\psi| < \sin^{-1} 1/r \\ &= n_2 \exp\left[-(t_2^{-1} - 1)s^2\right], \\ &\qquad |\psi| > \sin^{-1} 1/r. \end{aligned}\right\} \quad (13)$$

Using equations (10)–(12) we obtain

$$\left.\begin{aligned} n &= n_2 + \frac{1 - n_2}{\pi}\sin^{-1}\frac{1}{r} \\ nt &= n_2 t_2 + \frac{1 - n_2 t_2}{\pi}\sin^{-1}\frac{1}{r} \\ rQ &= (\sqrt{\pi})(1 - n_2 t_2^{\frac{3}{2}}) \\ n_2 &= \frac{1}{\sqrt{t_2}}. \end{aligned}\right\} \quad (14)$$

NUMERICAL PROCEDURES

Since equations (8) and (9) are similar it suffices to discuss just the numerical procedures used in determining g. First we observe that equation (8) is a partial differential equation. It thus appears that we must use interpolations to obtain terms like $u(x_0)$ as mentioned previously. However, because of the symmetry of the problem such interpolations can actually be avoided. We have made full use of this fact.

Following von Neumann's suggestion [14] we define two new variables $x = r\cos\psi$, $y = r\sin\psi$.

We note that y = constant is nothing but a characteristic line of equation (8). In terms of x and y (8) assumes the form

$$\begin{aligned} s\frac{\partial g}{\partial x} &= \alpha v[G - g] \\ G &= nt^{-1}\exp\left[-(t^{-1} - 1)s^2\right]. \end{aligned} \quad (15)$$

Equation (15) holds in an annulus $1 < x^2 + y^2 < b^2$ (Fig. 2) and the boundary conditions for g are

$$\begin{aligned} g &= 1, \quad x^2 + y^2 = 1, \quad x > 0 \\ g &= n_2 t_2^{-1}\exp\left[-(t_2^{-1} - 1)s^2\right], \\ &\qquad x^2 + y^2 = b^2, \quad x < 0. \end{aligned} \quad (16)$$

We also know that g is even in y. Because of the nature of the boundary conditions there are three kinds of integration paths for equation (15) as shown in Fig. 2. Physically a typical I-path corresponds to the free molecular trajectory of a particle emitted from the outer cylinder and ending up on the inner cylinder.

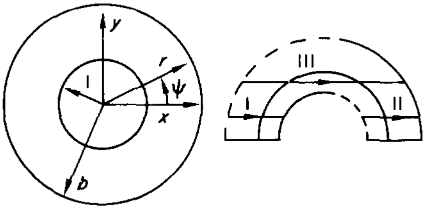

FIG. 2. Transformed-variable space.

In [14] an equation similar to equation (15) was integrated by converting the derivative term to a difference ratio. As pointed out previously because of the special form of the right-hand side of equation (15), it would be advantageous to integrate it first and then apply the finite difference scheme. To simplify our discussion we will choose $v = 1$, which was the case that Anderson considered [10]. We then integrate equation (15) to obtain

$$g(x, y, s) = g(x_0, y, s) \exp\left[-\frac{\alpha}{s}(x - x_0)\right] + \frac{\alpha}{s}\int_{x_0}^{x} \mathrm{d}x^1\, G(x^1, y, s) \exp\left[-\frac{\alpha}{s}(x - x^1)\right]. \qquad (17)$$

The initial values of g for each path of integration are given by equation (16). We are now ready to apply the discrete ordinate approximation and the finite difference approximation. Recall that both g and h are still distribution functions. What we are interested in are really the macroscopic quantities, i.e. integrals of g and h over the velocity space for fixed position r in physical space. Our numerical scheme should be devised accordingly.

We observe from equation (15) that s and y appear only as parameters. From the boundary conditions we expect that g will be bounded everywhere for any value of s. From the way that g is related to the macroscopic quantities such as in equation (10) we are motivated to use the modified half-range Gauss–Hermite quadrature points [13] for the s-variable. Next come the choice x and y, or, equivalently, r and ψ. In physical space, i.e. the r-coordinate, we would like to distribute the sample points so as to anticipate the behaviour of the macroscopic variables. For example, close to the inner cylinder n and t both have a sharp gradient in the near free-molecule flow as is evident from equation (14). As the Knudsen number gets smaller, n and t both have a boundary-layer type behaviour within a distance of the order of a mean free path from either boundary, known as the Knudsen layers. The boundary-layer behavior is milder in the outer Knudsen layer since the inner cylinder has smaller area than the outer one. Consequently, we would like to have the smallest increment of Δr close to the inner cylinder, the next larger close to the outer cylinder, and the largest in between. We can thus draw a series of concentric circles between the boundaries in the (x, y)-plane. The step-size of the x-integration, i.e. $\Delta x = x - x_0$, then varies from point to point. It is essentially the segment along any path line that lies in between two neighboring circles. It is evident that the choice of maximum allowable Δx also has an effect on the choice of Δr. Finally we note that the increment in y is determined by the maximum allowable increment for the ψ-variable at those r's at which we wish to evaluate the macroscopic quantities. Unlike the s variable the number of sample points for ψ varies for each given r. Since the free-molecular distribution function is discontinuous at $y = 1$ [see equation (13)] we will use a smaller increment in y near that point.

Once the sample points are chosen we are ready to solve equation (17) numerically. The structure of equation (17) suggests that an iteration scheme should be used. Let us write g^m, etc., as the solution of the mth iteration, and define

$$\begin{aligned} g^m_{ijk} &= g^m(x_i, y_j, s_k) \\ \Delta x_i &= x_{i+1} - x_i. \end{aligned} \qquad (18)$$

The iterative equation corresponding to equation (19) can be written as

$$g^m_{ijk} = g^m_{(i-1)jk} \exp\left[-\frac{\alpha}{s_k}\Delta x_{i-1}\right] + \frac{\alpha}{s_k}\int_{x_{i-1}}^{x_i} \mathrm{d}x^1\, G^{m-1}(x^1, s_k) \exp\left[-\frac{\alpha}{s_k}(x_i - x^1)\right]. \qquad (19)$$

The simplest way to evaluate the integral is perhaps to replace G^{m-1} by

$$\bar{G}^{m-1}_{ik} = \tfrac{1}{2}\left[G^{m-1}(x_i, s_k) + G^{m-1}(x_{i-1}, s_k)\right] \qquad (20)$$

and equation (19) then becomes

$$g^m_{ijk} = g^m_{(i-1)jk} \exp\left[-\frac{\alpha}{s_k}\Delta x_{i-1}\right] + \bar{G}^{m-1}_{ik}\left[1 - \exp\left(-\frac{\alpha}{s_k}\Delta x_{i-1}\right)\right]. \qquad (21)$$

For fixed Δx_i the above approximation is obviously not a good one when α is large, i.e. towards the continuum limit. The integral in equation (19) has the form of a Laplace integral. From the asymptotic expansion of such an integral for large we know that it should be dominated by the behavior of G^{m-1} at $x = x_1$. In fact the asymptotic expansion of equation (17) for large α and finite s is

$$g(x, y, s) = G(x, s) + \frac{s}{\alpha}\frac{\partial G}{\partial x} + O\left(\frac{1}{\alpha^2}\right) + (\text{polynomials of } \frac{1}{\alpha}) \exp\left[-\frac{\alpha}{s}(x - x_0)\right]. \qquad (22)$$

Physically this is because the characteristics of equation (8) are nothing but the free-molecular trajectories. Initial information cannot be carried too far down such lines due to numerous collisions along the way. The simplest way to remedy this seems to be to just decrease the size of the increment Δx_i. One may also use better interpolation schemes than that of equation (20). Another way would be to devise an approximation scheme based on the asymptotic form equation (22). However, no such attempts were made in the last two directions since our primary interest is not to generate precise numerical numbers.

To start an iteration we adopt the strategy used by Anderson [10]. The free molecular solution is used to start the iteration for $\alpha = 0{\cdot}01$. The final result of $\alpha = 0{\cdot}01$ then is used as the first guess for $\alpha = 1$. The process is repeated for increasing α.

RESULTS AND DISCUSSIONS

Computations have been carried out on a CDC-6400 computer for a number of cases with $b = 2{\cdot}0$. The bulk of the calculations were performed assuming constant collision frequency, i.e. $\nu = 1$, and with temperature ratio $t_2 = 0{\cdot}909$, 0·5 and 0·25. Some calculations have also been done for $\nu = n$, $nT^{\frac{1}{2}}$ and $nT^{1-0.816}$ which correspond to the collision frequencies of a Maxwell molecule gas, a hard-sphere molecule gas, and argon gas, respectively [15]. All the results presented in the following are based on 16 sample points in the r-variable, 6 to 21 in the ψ-variable depending on r, and 4 to 6 in the s-variable depending on the temperature ratio t_2. How these sample points are chosen was described in the last section. With such a choice of sample points each iteration takes slightly less than 1·3 s. For each α the iterative procedure is terminated if both the relative difference of $Q(r = 1)$ for successive iterations is less than 0·001 and the values $Q_t = rQ$ for all r differ less than 5 per cent. In fact except for α close to 10 the defect of Q_T is usually less than 2 per cent. It may be pointed out that the numerical quadrature that we used yields $Q(r = 1) = 0{\cdot}8821$ for free molecular case while the exact value should be 0·8862. Typically we need one iteration for $\alpha = 0{\cdot}01$ and about 20 iterations for $\alpha = 1$ to 10. Based on the results the following statements can be made.

(1) To test the numerical algorithm we have run cases with $t_2 = 1{\cdot}0$, i.e. the trivial case, for $\nu = 1$ and $\alpha = 0{\cdot}01$–$8{\cdot}0$. In these cases both n and t should be equal to unity everywhere and there should be no net heat flux, i.e. $Q_T = 0$. It is found that both n and t are correct to the 7th digit when $\alpha = 1{\cdot}0$ and 5th when $\alpha = 8{\cdot}0$. Both Q_T and M are found to be on the order of 10^{-7}.

(2) The number density n and temperature t are plotted vs. r in Figs. 3–6 for $t_2 = 0{\cdot}5$ and 0·25 and for $\alpha = 0{\cdot}01$, 1·0, 3·0 and 10·0. All of these are for $\nu = 1$. It can be seen that the results are qualitatively the same as the linear case. Thus the non-linearity introduces complexity into the mathematics, but does not drastically alter the behavior. Anderson [10] has also treated some cases with $t_2 = 0{\cdot}909$ and 0·5. He used an integral equation approach which is appropriate only to the BGK-model equation. Our results checked quite well with his. There are only slight deviations from his curve in the center region. He used a parameter λ in place of the α that we are using. They are related by $\lambda = (t_2/2)^{\frac{1}{2}}\, 1/\alpha$. The results for the total heat flux

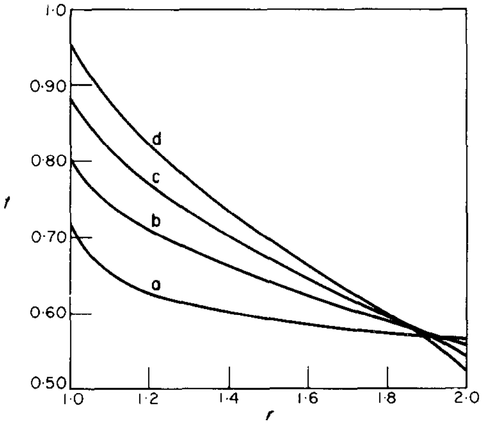

FIG. 3. Temperature vs. r [$\nu = 1$; $t_2 = 0{\cdot}5$; $\alpha = 0{\cdot}01$ (ε), 1·0 (b) 3·0 (c), 10·0 (d)].

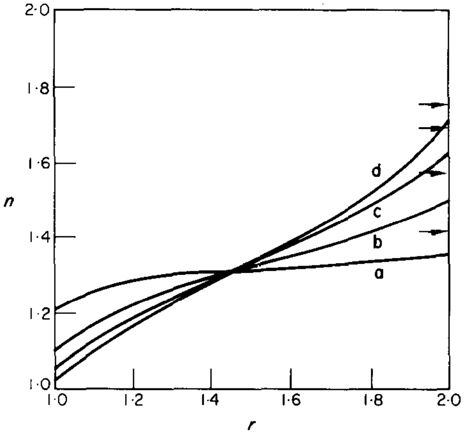

FIG. 4. Number density vs. r [$\nu = 1$; $t_2 = 0{\cdot}5$; $\alpha = 0{\cdot}01$ (a), 1·0 (b), 3·0 (c), 10·0 (d); → denotes the corresponding n_2 values].

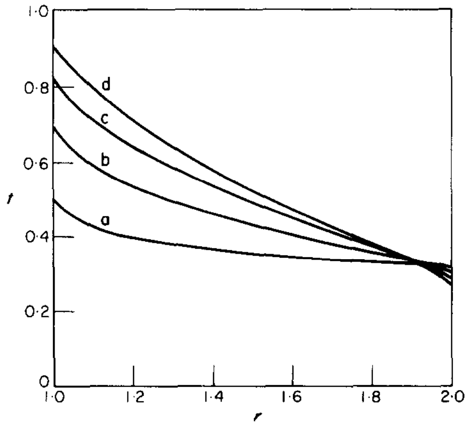

FIG. 5. Temperature vs. r [$\nu = 1$; $t_2 = 0{\cdot}25$; $\alpha = 0{\cdot}01$ (a), 1·0 (b), 3·0 (c), 10·0 (d)]

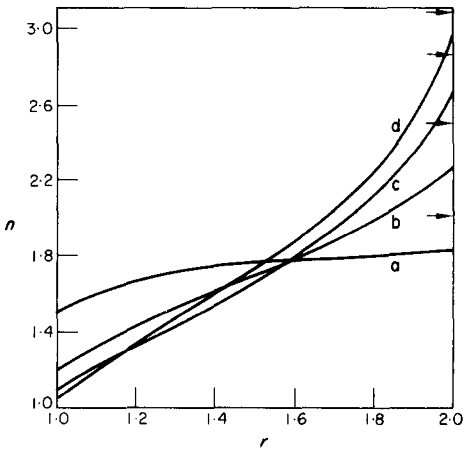

FIG. 6. Number density vs. r [$\nu = 1$; $t_2 = 0{\cdot}25$; $\alpha = 0{\cdot}01$ (a), 1·0 (b), 3·0 (c), 10·0 (d); → denotes the corresponding n_2 values].

normalized by the free molecular total heat flux and n_2 are plotted vs. $1/\alpha$ in Figs. 7 and 8. In the dimensional case the free-molecular heat flux depends on the temperature difference. Since the continuum case shows this same dependence we would expect no great deviation from it for intermediate regimes and, thus, the ratio exhibits the insensitivity to t_2 shown. This is also confirmed by Anderson's results. n_2, however, depends significantly on t_2. Since n_2 is related to the half-range mass flux at the outer wall this is as expected. We believe that there is a typographical error in Anderson's paper since our heat flux curve falls right on his curve for the spherical case and not the cylindrical one. From physical considerations for the same α the

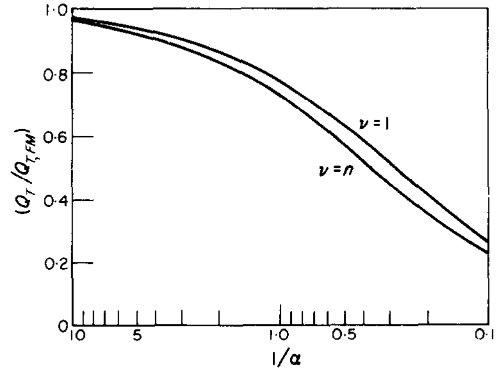

FIG. 7. Heat flux vs. $1/\alpha$.

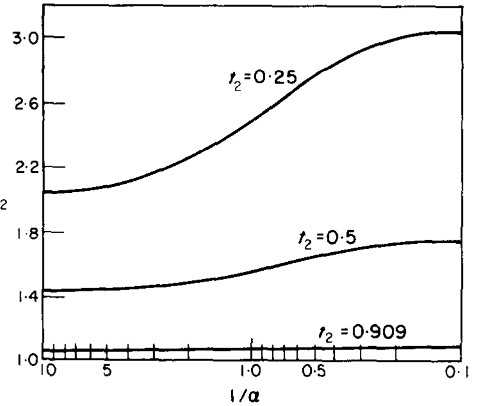

FIG. 8. n_2 vs. $1/\alpha$.

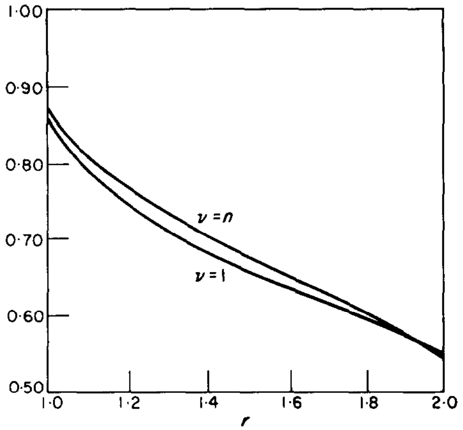

FIG. 9. Temperature vs. r ($t_2 = 0·5$; $\alpha = 2·0$).

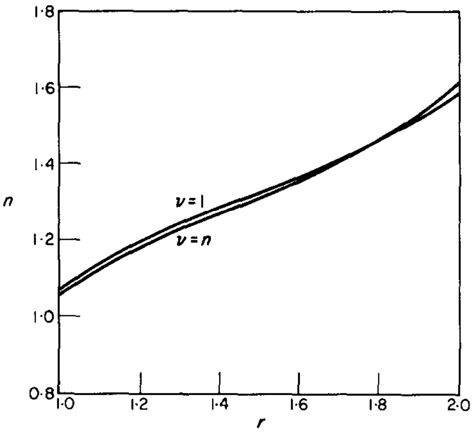

FIG. 10. Number density vs. r ($t_2 = 0·5$; $\alpha = 2·0$).

cylindrical heat transfer should be larger than the spherical one and the reverse is true as represented by Anderson's figures.

(3) As pointed out previously, another check for the numerical procedure is that M should vanish identically, see equation (12). For $t_2 = 0·909$ case the final values of M are less than 0·001 while for $t_2 = 0·5$ and 0·25 they are less than 0·0075 and 0·02 respectively. More iterations tend to drive M towards zero, but the rate is very slow. By decreasing the step-size, or by having smaller increments of y around $y = 1$, or by increasing the number of sample points in s we can improve the results, but the improvement is not comparable to the effort. It is believed that the cause of this slow convergence is mainly due to the crude approximation used in equation (20).

(4) To study the influence of the variable collision frequency we have also carried out similar calculations for $\nu = n$, $nT^{\frac{1}{2}}$ and $nT^{1-0·816}$. There are only slight changes in the number density and temperature profiles. In Figs. 9 and 10 n and t are plotted vs. r for $t_2 = 0·5$. Only curves corresponding to $\nu = 1$ and $\nu = n$ are plotted; those for the other two values of ν lie between them and are hence omitted for clarity. An increased collision frequency causes the flow to behave more nearly like a contiuum. Thus the temperature is closer to one at the inner cylinder and closer to 0·5 at the outer cylinder for $\nu = n$. The smallness of the change is due to the fact that the temperature and density changes are in opposite directions. Thus, for example, the temperature increase at the inner cylinder is accompanied by a density, and hence collision frequency, decrease thus minimizing the effect. The heat flux curves are different for different t_2's as they should be. The heat flux curve for $\nu = n$ and $t_2 = 0·5$ is plotted in Fig. 7. The curves for $\nu = nT^{\frac{1}{2}}$ and $\nu = nT^{1-0·816}$ lie in between those of $\upsilon = 1$ and $\nu = n$. For given t_2 there are only slight changes in the values of n_2 for all the ν's.

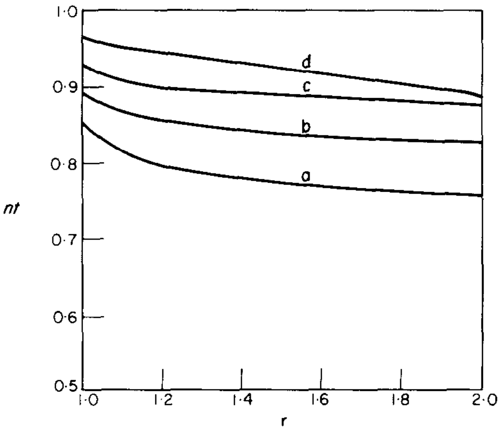

FIG. 11. (Number density) × (Temperature) vs. r [$\nu = 1$; $t_2 = 0{\cdot}5$; $\alpha = 0{\cdot}01$ (a), $1{\cdot}0$ (b), $3{\cdot}0$ (c), $10{\cdot}0$ (d).

(5) It is known that in the continuum limit $p = nt =$ constant. Therefore we can use the product nt as a measure of the deviation from continuum. In Fig. 11 we have plotted nt vs. r for $\nu = 1$, $t_2 = 0{\cdot}5$ and $\alpha = 0{\cdot}01$, 1·0, 3·0 and 10·0. We note that the spread of each curve decreases as α increases for the first three curves, as one might expect from physical intuition. However, the spread for the curve $\alpha = 10{\cdot}0$ is wider than that of $\alpha = 3{\cdot}0$. This perhaps is due to the inherent defect in the numerical algorithm, depending on b, α, t_2, the interpolation scheme equation (20), and the sample points used (see the following).

(6) To test the dependence of the solution on the initial guess and the convergence of the numerical algorithm we have run 250 iterations for $\alpha = 1{\cdot}0$ and 600 iterations for $\alpha = 8{\cdot}0$, both for $t_2 = 0{\cdot}5$ and $\nu = 1$. In both cases we started with the free-molecular solution as the first guess. At the beginning the iteration converges relatively fast and the defect for Q_T, i.e. the relative spread of Q_T, soon reaches a minimum. Afterwards, however, the defect of Q_T increases gradually and attains some limiting value which should depend on b, t_2, α, the interpolation scheme equation (20), and the choice of sample points. For example, in the $\alpha = 1{\cdot}0$ case the minimum was assumed at 1·77 per cent after 8 iterations, then rose gradually to 2·10 per cent at the end of the 48th iteration, and stayed just above 2·125 per cent after 102 iterations. For $\alpha = 8{\cdot}0$ the defect of Q_T assumes the minimum 1·57 per cent at the 60th iteration and rises to just below 5·77 per cent at the end of 500th run, while $Q_T(r = 1)/(Q_T)_{FM}$ changes from 0·319 to 0·300. In principle one could also use the extrapolation technique used by Anderson [16] to accelerate the convergence of the iteration scheme. A better interpolation scheme than that of equation (20) will also improve the convergence as discussed previously, especially when α is large. However, no such attempts were made.

(7) As pointed earlier the numerical scheme that we are using is not expected to be good near the continuum limit. The same difficulty is also shared by other discrete ordinate methods [4] and the integral equation method [10]. In [9] we found that by substracting the continuum solution from the distribution function we can obtain better solutions towards the continuum end. However, for the present problem and for the cases that we treated such a modification is not needed.

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UNE MÉTHODE ORDONÉE DISCRÈTE MODIFÉE POUR TRAITER LE TRANSFERT THERMIQUE CYLINDRIQUE NON LINÉAIRE

Résumé—Le transfert de chaleur non linéaire entre des cylindres concentriques est étudié numériquement à partir du modèle B.G.K. Le procédé numerique adopté est une combinaison de la procédure itérative intégrale de Willis et de la méthode ordonnée discrète. Les résultats, pour une fréquence de collision constante, sont très proches d'autres résultats numériques, obtenus à partir de l'approche de l'équation intégrale.

Des cas de fréquence de collision variable sont aussi considérés et les résultats montrent qu'il n'y a que de légers changements par rapport aux cas de fréquence de collision rapide.

La méthode utilisée dans cet article peut-être appliquée à une large variété de problèmes.

ANWENDUNG DES VERFAHRENS DER DISKRETEN ORDINATE AUF DEN WÄRMEÜBERGANG BEI ZYLINDERGEOMETRIE

Zusammenfassung— Unter Anwendung der BGK-Modell-Gleichung wird der nichtlineare Wärmeübergang zwischen konzentrischen Zylindern numerisch untersucht.

Das verwendete numerische Verfahren ist eine Kombination des Willisschen-Integral-Iterations-Schemas und der Methode der diskreten Ordinate. Die Ergebnisse für konstante Kollisionsfrequenz stimmen mit bekannten numerischen Lösungen, nach der Integral-Methode, sehr gut überein. Fälle mit variabler Kollisionsfrequenz wurden untersucht, und die Ergebnisse zeigen, dass hier nur unbedeutende Veränderungen mit Rücksicht auf die Fälle augenblicklicher Kollisionsfrequenz auftreten. Die in dieser Arbeit verwendete Methode kann auf eine grosse Klasse von Problemen angewandt werden.

ПРИМЕНЕНИЕ МОДИФИЦИРОВАННОГО ДИСКРЕТНОГО МЕТОДА ОРДИНАТ ДЛЯ СЛУЧАЯ НЕЛИНЕЙНОГО ТЕПЛООБМЕНА ЦИЛИНДРА

Аннотация—Проводится численное исследование нелинейного теплообмена между концентрическими цилиндрами с помощью модельного уравнения *BGK*. Расчет проводится путем совместного применения интегральной итерационной схемы Виллиса и метода дискретных ординат. Сравнение результатов для постоянной частоты столкновений показало хорошее согласование с другими имеющимися численными данными, полученными методом интегрального уравнения. Кроме того, рассматриваются случаи переменной частоты столкновений. Показано, что результаты слабо отличаются от данных при мгновенных скоростях столкновений. Используемый метод применим для большого класса задач.